

Tiling Periodicity

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The string S has a **full period** if

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Equivalently,

$$\forall 1 \leq i < i + p \leq n : s_i = s_{i+p}$$

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Not in the classical sense. But...

Outline of the Talk

- 1 Notion of Tiling Periodicity
- 2 Properties of Tiling Periodicity
- 3 Finding Tiling Periods of Minimal Size
- 4 Conclusions and Future Work

Notion of Tiling Periodicity

Motivating Examples

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The string above is not periodic, but the red **structure**

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is a kind of period, since we can cover initial string by **four parallel copies** of it:

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The simplest example:

A A B B

Formal Definition

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A tiling string S is called the **tiling period** of (ordinary) string T if we can cover T by parallel copies of S satisfying the following:

- All defined (visible) letters of S -copies match the text letters
- Every text letter covered by **exactly one** defined (visible) letter

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Conjecture: tiling periodicity is not expressible in word equations
- Relations to multidimensional periodicity
- Pattern discovery. At least my talk is in “pattern discovery” session :-)

Partial Order on Tilings

Definition: a tiling S is **smaller** than a tiling Q iff Q can be splitted into several parallel copies of S satisfying the following:

- All defined (visible) letters of S -copies match the visible Q letters
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is less than

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Primitive Tiling Period Conjecture

Conjecture: For every ordinary string there exists a unique primitive tiling period (it is less than any other tiling period).

Reformulation: Any two tiling periods have a common tiling “subperiod”

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Surprisingly, the conjecture is wrong! Look at the (minimal known) counterexample:

A A A A A A A A B A A B B A A B A A A A A A A

and

A A A A A A A A B A A B B A A B A A A A A A A

Properties of Tiling Periodicity

How Many Tiling Periods? (1/2)

Bodini & Rivals (CPM'06) studied number of tilings $L(n)$ of **unary** word of length n :

- $L(1) = 1$, for every $n > 1$

$$L(n) = 1 + \sum_{d|n, d \neq n} L(d)$$

- $L(36) = 52$

How Many Tiling Periods? (2/2)

Our result:

Theorem

There is one-to-one correspondence between tiling of unary word of length n and factorizations $n = n_1 \cdot \dots \cdot n_k$ where $n_2, \dots, n_k \geq 2$

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Theorem

Take any pair of tiling period and classical period. Then they have a common “tiling subperiod”

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Reformulation

Any primitive tiling period of string T is also a tiling period of any classical period of T

Finding Tiling Periods of Minimal Size

Auxiliary Definition: Multi-Period

A word has **multi-period** (a, b) iff $a|b$ and for every k a $[kb + 1, (k + 1)b]$ block has the full period a

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The text above has multi-periods $(1, 2)$ and $(4, 8)$

Definition: multi-period (a, b) is **embedded** into another one a', b' iff $b|a'$

Tiling Period and “his” Multi-Periods

Multi-Period Lemma

Every tiling period corresponds to some sequence of embedded multi-periods $(a_1, b_1) \dots (a_k, b_k)$. The size of period is equal $n \prod_{i=1}^k \frac{a_i}{b_i}$

Preprocessing Step

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There is $\mathcal{O}(n \log n \log \log n)$ preprocessing of the text such every query “is (a, b) a multi-period” can be answered in $\mathcal{O}(\log n)$ time

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Trick: Karp-Miller-Rosenberg algorithm

Finding Tiling Periods of Minimal Size

Theorem

There is $\mathcal{O}(n \log n \log \log n)$ algorithm for finding a tiling period of minimal size

Conclusions and Future Work

Directions for Further Research

- Whether all primitive tiling periods have the same number of visible letters?
- How often strings are tiling periodic?
- Introduce **not full** tiling periods. How to find the one of minimal size?
- Find natural sources of tiling periodicity

Summary

- New notion: tiling periodicity

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- **Result 3:** there is bijection between tiling periods of unary words and length factorizations

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- **Result 1:** the primitive tiling period is not necessary unique
- **Result 2:** tiling periods live “inside” classical
- **Result 3:** there is bijection between tiling periods of unary words and length factorizations
- **Result 4:** $\mathcal{O}(n \log n \log \log n)$ algorithm for tiling periods of minimal size

Last Slide

Search “**Lifshits**” or visit <http://logic.pdmi.ras.ru/~yura/>



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Thank you for your attention! Questions?