Tiling Periodicity

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Classical Notion of Periodicity

The string S is called **purely periodic** if

$$S = W^k = W \dots W$$

Equivalently

$$\forall \quad 1 \leq i < i + p \leq n : \quad s_i = s_{i+p}$$

Is the following string purely periodic?

A	A	B	B	A	A	B	B	C	C	D	D	D	C	C	D	D

Not in the classical sense. But...

Outline of the Talk

1. Notion of Tiling Periodicity
2. Minimal Tiling Period Conjecture
3. Properties of Tiling Periodicity
   - Maximal Number of Periods
   - Relation to Classical Periodicity
   - Algorithm for Finding Minimal Tiling Periods
4. Future Work

Motivating Examples

A	A	B	B	A	A	B	B	C	C	D	D	D	C	C	D	D

The string above is not periodic, but pink **structure**

A	A	B	B	A	A	B	B	C	C	D	D	D	C	C	D	D

is a kind of period, since we can cover initial string by **four parallel copies** of it:

A	A	B	B	A	A	B	B	C	C	D	D	D	C	C	D	D

The simplest example:

A	A	B	B

Formal Definition

A **tiling string** (or tiler) is a string over $\Sigma \cup \square$ alphabet, where $\square$ is a special **transparent** (or undefined) letter. Sometimes the term **partially defined word** is also used

A tiling string S is called the **tiling period** of (ordinary) string T if we can cover T by parallel copies of S satisfying the following:

- All defined (visible) letters of S-copies match the text letters
- Every text letter covered by **exactly one** defined (visible) letter

Why Tiling Periodicity

- New structural properties of texts (Conjecture: tiling periodicity is not expressible in word equations)
- New tool for text compression
- Relations to multidimensional periodicity
- Natural generalization of the classical notion
- Pattern discovery (?????)

Partial Order on Tilers

We say that one tiling string (tiler) S is **smaller** than another tiler Q, if Q can be covered by several parallel copies of S satisfying the following:

- All defined (visible) letters of S-copies match the visible Q letters
- Every Q letter covered by **exactly one** defined (visible) letter

Example:

A	A	B	B	A	A	B	B	C	C	D	D	D	C	C	D	D

is less than

A	A	B	B	A	A	B	B	C	C	D	D	D	C	C	D	D

Minimal Tiling Period Conjecture

Main Conjecture: For every ordinary string there exists a unique minimal tiling period (it is less than any other tiling period).

Reformulation Any two tiling periods have a common tiling "subperiod"

Big surprise (at least for me): conjecture is wrong! Look at (minimal known) counterexample:

A	A	A	A	A	A	A	A	B	A	A	B	B	A	A	B	A	A	A	A	A	A	A	A

and

A	A	A	A	A	A	A	A	B	A	A	B	B	A	A	B	A	A	A	A	A	A	A	A

How Many Tiling Periods?

Let $L(n)$ be the number of periods of the string of length n over a unary alphabet. Then:

- L(1)=1
- For every $n > 1$ we can compute L by recursive formula:

$$L(1) = 2; L(n) = \sum_{d|n, d \neq n} L(d)$$

- $L(36) = 52$
- $L(p_1 \cdot \dots \cdot p_k) = (k+1)!$
- (To be done) What is the upper limit of $L(n)/n$?

Tiling Periods are Always Smaller than Classical

Theorem Take any pair of tiling period and classical period. Then they have a common "tiling subperiod". Any minimal tiling period of string T is also a tiling period of any classical period of T.

Finding Minimal Tiling Periods: Sketch

1. Define a notion of "ranged periodicity"
2. Prove that any minimal tiling root corresponds to the "best" chain of embedded ranged periodicities
3. Find all ranged periodicities
4. Find the "best" chain

Directions for Further Research

- Study **not pure** tiling periodicity
- How often strings are tiling periodic?
- Whether the property **"string has a tiling square root"** can be expressed by word equations?
- Whether all minimal tiling roots have the same number of visible letters?
- Find natural sources of tiling periodicity
- Improve the complexity of the algorithm for finding minimal tiling periods
- Find relevant references

Summary

Main points:

- New notion: tiling periodicity

A	A	B	B	A	A	B	B	C	C	D	D	C	C	D	D

- The minimal tiling root is not necessary unique!
- Algorithm for finding minimal tiling roots

Last Slide

Related paper will appear soon at
http://logic.pdmi.ras.ru/~yura/

Thanks for inviting to the seminar
Thanks for opportunity for the second talk
Thanks for attention

Questions?