

# Tiling Periodicity

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# Classical Notion of Periodicity

The string  $S$  is called **purely periodic** if

$$S = W^k = W \dots W$$

Equivalently

$$\forall 1 \leq i < i + p \leq n : s_i = s_{i+p}$$

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Is the following string purely periodic?

A	A	B	B	A	A	B	B	C	C	D	D	C	C	D	D
---	---	---	---	---	---	---	---	---	---	---	---	---	---	---	---

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Not in the classical sense. But...

# Outline of the Talk

- 1 Notion of Tiling Periodicity
- 2 Minimal Tiling Period Conjecture
- 3 Properties of Tiling Periodicity
  - Maximal Number of Periods
  - Relation to Classical Periodicity
  - Algorithm for Finding Minimal Tiling Periods
- 4 Future Work

# Motivating Examples

A A B B A A B B C C D D C C D D

The string above is not periodic, but pink **structure**

A A B B A A B B C C D D C C D D

is a kind of period, since we can cover initial string by **four parallel copies** of it:

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The simplest example:

A A B B

# Formal Definition

A **tiling string** (or tiler) is a string over  $\Sigma \cup \square$  alphabet, where  $\square$  is a special **transparent** (or undefined) letter. Sometimes the term **partially defined word** is also used



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A tiling string  $S$  is called the **tiling period** of (ordinary) string  $T$  if we can cover  $T$  by parallel copies of  $S$  satisfying the following:

- All defined (visible) letters of  $S$ -copies match the text letters
- Every text letter covered by **exactly one** defined (visible) letter

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- New tool for text compression
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- Natural generalization of the classical notion
- Pattern discovery (?????)

# Partial Order on Tilers

We say that one tiling string (tiler)  $S$  is **smaller** than another tiler  $Q$ , if  $Q$  can be covered by several parallel copies of  $S$  satisfying the following:

- All defined (visible) letters of  $S$ -copies match the visible  $Q$  letters
- Every  $Q$  letter covered by **exactly one** defined (visible) letter

**Example:**

A	A	B	B	A	A	B	B	C	C	D	D	C	C	D	D
---	---	---	---	---	---	---	---	---	---	---	---	---	---	---	---

is less than

A	A	B	B	A	A	B	B	C	C	D	D	C	C	D	D
---	---	---	---	---	---	---	---	---	---	---	---	---	---	---	---

# Minimal Tiling Period Conjecture

**Main Conjecture:** For every ordinary string there exists a unique minimal tiling period (it is less than any other tiling period).

**Reformulation** Any two tiling periods have a common tiling “subperiod”



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**Reformulation** Any two tiling periods have a common tiling “subperiod”

Big surprise (at least for me): conjecture is wrong! Look at (minimal known) counterexample:

A	A	A	A	A	A	A	A	B	A	A	B	B	A	A	B	A	A	A	A	A	A	A	A
---	---	---	---	---	---	---	---	---	---	---	---	---	---	---	---	---	---	---	---	---	---	---	---

and

A	A	A	A	A	A	A	A	B	A	A	B	B	A	A	B	A	A	A	A	A	A	A	A
---	---	---	---	---	---	---	---	---	---	---	---	---	---	---	---	---	---	---	---	---	---	---	---

# How Many Tiling Periods?

Let  $L(n)$  be the number of periods of the string of length  $n$  over a unary alphabet. Then:

- $L(1)=1$
- For every  $n > 1$  we can compute  $L$  by recursive formula:

$$L(1) = 2; L(n) = \sum_{d|n, d \neq n} L(d)$$

- $L(36) = 52$
- $L(p_1 \cdots p_k) = (k + 1)!$
- (To be done) What is the upper limit of  $L(n)/n$ ?

# Tiling Periods are Always Smaller than Classical

**Theorem** Take any pair of tiling period and classical period. Then they have a common “tiling subperiod”. Any minimal tiling period of string  $T$  is also a tiling period of any classical period of  $T$ .

# Finding Minimal Tiling Periods: Sketch

- 1 Define a notion of “ranged periodicity”
- 2 Prove that any minimal tiling root corresponds to the “best” chain of embedded ranged periodicities
- 3 Find all ranged periodicities
- 4 Find the “best” chain

# Directions for Further Research

- Study **not pure** tiling periodicity
- How often strings are tiling periodic?
- Whether the property **“string has a tiling square root”** can be expressed by word equations?
- Whether all minimal tiling roots have the same number of visible letters?
- Find natural sources of tiling periodicity
- Improve the complexity of the algorithm for finding minimal tiling periods
- Find relevant references

# Summary

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- Algorithm for finding minimal tiling roots



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**Questions?**