New Algorithms on Compressed Texts

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FCPM: Problem Description

Fully Compressed Pattern Matching (FCPM)

INPUT: Compressed strings P and T
OUTPUT: Yes/No (whether P is a substring in T?)

Example

Text: abaababaabaab

Pattern: **baba**

We know only compressed representation of P and T

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Outline of the Talk

1. Processing Compressed Texts: Bird's Eye View
2. Fully Compressed Pattern Matching: Idea of a New Algorithm
   - Idea of a new algorithm
   - ★ Detailed description
3. More Algorithms and Some Negative Results
4. Conclusions and Open Problems

Processing Compressed Text

Central idea

If some text is highly compressible,
then it contains long identical segments
and therefore it is likely that we can solve
some problems more efficiently than in general case

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Reasons for algorithms on compressed texts:

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Real data with high level of repetitions:

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- Automatically generated texts

Straight-Line Programs

Straight-line program (SLP) is a

- Context-free grammar generating **exactly one** string
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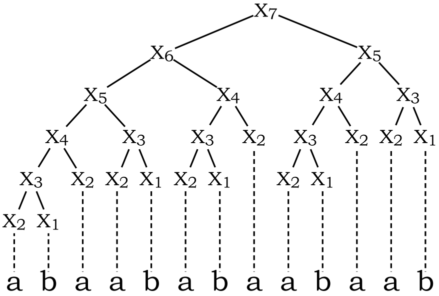
Example

abaababaabaab

$$X_1 \longrightarrow b, \qquad X_2 \longrightarrow a$$
$$X_3 \longrightarrow X_2X_1, \quad X_4 \longrightarrow X_3X_2$$
$$X_5 \longrightarrow X_4X_3, \quad X_6 \longrightarrow X_5X_4$$
$$X_7 \longrightarrow X_6X_5$$



Important Related Results

Algorithms on compressed texts:

- **Amir et al.'94:** Compressed Pattern Matching
- **Gąsieniec et al.'96:** Regular Language Membership

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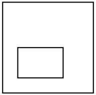
The following problems are hard for compressed texts:

- **Lohrey'04:** Context-Free Language Membership
- **Berman et al.'02:** Two-dimensional Compressed Pattern Matching



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Lifshits'06: $O(n^2 m)$ algorithm

Basic Lemma

Notation:

Position = place between neighbor characters.
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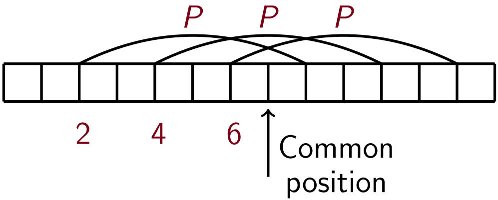
Lemma

All occurrences of P in T touching any given position form a single arithmetical progression



AP-table

Let $P_1, \ldots, P_m$ and $T_1, \ldots, T_n$ be the compression symbols.

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AP-table:

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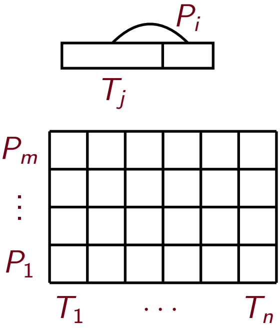
let $AP[i,j]$ be a code
of ar.pr. of occurrences of P_i in T_j
that touches the cut of T_j



Two Claims

Claim 1: We can solve all variants of FCPM from AP-table in linear time:

- Find the first occurrence
- Count the number of all occurrences
- Check whether there is an occurrence from the given position
- Compute a “compressed” representation of all occurrences

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- Find the first occurrence
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Claim 2: We can compute the whole AP-table by dynamic programming method using $O(n)$ time for every element

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AP-table:

for every $1 \leq i \leq m, 1 \leq j \leq n$ let $AP[i,j]$ be a code of ar.pr. of occurrences of P_i in T_j that touches the cut of T_j

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Answer:
P occurs in T iff there is j such that $AP[m, j]$ is nonempty

Computing AP-table

Order of computation:

```
from j=1 to n do
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We design a special auxiliary procedure that extracts useful information from already computed part of AP-table for computing a new element $AP[i, j]$

Auxiliary Procedure: Local PM

$\mathit{LocalPM}(i, j, [\alpha, \beta])$ returns occurrences of P_i in T_j inside the interval $[\alpha, \beta]$

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Important properties:

- Local PM uses values AP[i,k] for $1 \leq k \leq j$
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Proposition: answer of Local PM indeed could be always represented by pair of ar.pr.

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Remark: we can do only step 1 by Local PM
Idea: not all occurrences of P_s but only these that are starting at the ends of P_r ones.

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Some blackboard explanation...

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Total complexity:

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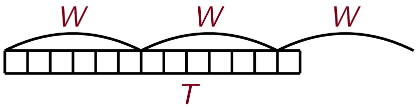
We are done! (Modulo basic computation of AP-table and realization of Local PM)

Covers and Periods

A **period** of a string T is a string W such that T is a prefix of W^k for some integer k



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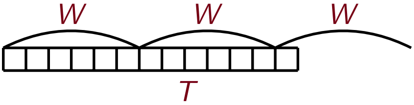
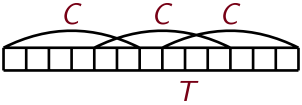


A **cover** of a string T is a string C such that any character in T is covered by some occurrence of C in T



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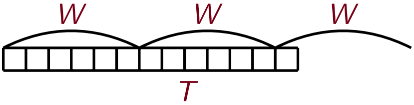

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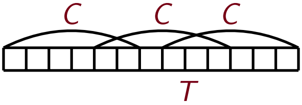

Compressed Periods/Covers: given a compressed string T, to find the shortest period/cover and compute a "compressed" representation of all periods/covers

Subsequence Problems

Compressed Window Subsequence: given a pattern P, a compressed string T, and an integer k, to determine whether P is a **scattered** subsequence in some window of length k in the text T

Example

T: abaababaabaab

P: babab

k : 6

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Hamming Distance and LCS

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T_1: a**baababaabaab** $LCS(T_1, T_2) = 12$
T_2: **baababa**b**abaab**

Fingerprint Table

A **fingerprint** is a set of used characters of any substring of T. A **fingerprint table** is the set of all fingerprints.

Example

Text: abacaba

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Compressed Fingerprint Table: given a compressed string T, to compute a fingerprint table

Check Your Intuition

Which of the following problems have polynomial algorithms?

1. Periods
2. Longest Common Subsequence
3. Hamming distance
4. Covers
5. Fingerprint Table
6. Compressed Window Subsequence
7. Fully Compressed Subsequence Problem

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Answer: **red-on-grey** problems have polynomial algorithms, black ones are NP-hard

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- New algorithm: fully compressed pattern matching in cubic time
- More algorithms: covers, periods, window subsequence, fingerprint table. But LCS, Hamming distance, FCSP are NP-hard.

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Main points:

- New field: algorithms working on compressed objects (including strings) without unpacking them
- New algorithm: fully compressed pattern matching in cubic time
- More algorithms: covers, periods, window subsequence, fingerprint table. But LCS, Hamming distance, FCSP are NP-hard.

Open Problems

- To construct a $O(nm \log |T|)$ algorithm for Fully Compressed Pattern Matching
- To construct $O(nm)$ algorithms for edit distance, where n is the length of T_1 and m is the **compressed size** of T_2

Last Slide

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- Yu. Lifshits
  Solving Classical String Problems on Compressed Texts.
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Thanks for attention.

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Thanks for attention. **Questions?**